Math 31 – Homework 7

Note: This assignment is optional.

Note: Any problem labeled as "show" or "prove" should be written up as a formal proof, using complete sentences to convey your ideas.

Basic Ring Theory

The problems on this list all involve basic definitions and examples of rings, along with ring homomorphisms. You should be able to do them all after the x-hour on August 13.

1. Let R be an integral domain. If $a, b, c \in R$ with $a \neq 0$ and $ab = ac$, show that $b = c$.

2. Find the following products of quaternions.

(a) $(i + j)(i - j)$.

(b) $(1 - i + 2j - 2k)(1 + 2i - 4j + 6k)$.

(c) $(2i - 3j + 4k)^2$.

(d) $i(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k) - (\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)i$.

3. Let R be a commutative ring with identity. Show that if $u \in R$ is a unit, then u is not a zero divisor. Conclude that any field is necessarily an integral domain. [**Note:** This is proven in Corollary 16.3 of Saracino if you'd like to check your answer there.]

4. Let R be a finite integral domain with identity $1 \in R$. Show that R is actually a field. [**Note:** This is Theorem 16.7 in Saracino.]

5. [Saracino, #16.16] Let R be a ring. An element $r \in R$ is a (multiplicative) **idempotent** if $r^2 = r$. We say that R is a **Boolean ring** if every element of R is a multiplicative idempotent. If R is Boolean, show that

(a) $2r = 0$ for every $r \in R$ (i.e., $r = -r$).

(b) R is commutative.

6. Let R and S be two rings with identity, and let 1_R and 1_S denote the multiplicative identities of R and S, respectively. Let $\varphi : R \to S$ be a nonzero ring homomorphism. (That is, φ does not map every element of R to 0.)

(a) Show that if $\varphi(1_R) \neq 1_S$, then $\varphi(1_R)$ must be a zero divisor in S. Conclude that if S is an integral domain, then $\varphi(1_R) = 1_S$.

(b) Prove that if $\varphi(1_R) = 1_S$ and $u \in R$ is a unit, then $\varphi(u)$ is a unit in S and

$$\varphi(u^{-1}) = \varphi(u)^{-1}.$$

Ideals and Polynomials

The following questions deal with ideals, quotient rings, and polynomial rings. You should be able to complete them after class on Monday, August 19.

1. Let R be a ring, and suppose that I and J are ideals in R. Prove that $I \cap J$ is an ideal in R.

2. Let R be a commutative ring. An element $a \in R$ is said to be **nilpotent** if there is a positive integer n such that $a^n = 0$. The set

$$\mathrm{Nil}(R) = \{a \in R : a \text{ is nilpotent}\}$$

is called the **nilradical** of R. Prove that the nilradical is an ideal of R. [**Hint:** You may need to use the fact that the usual binomial theorem holds in a commutative ring. That is, if $a, b \in R$ and $n \in \mathbb{Z}^+$, then

$$(a+b)^n = \sum_{k=0}^{n} a^{n-k} b^k.$$

This should help with checking that $\mathrm{Nil}(R)$ is closed under addition.]

3. [Saracino, #17.14] Let R be a ring and I an ideal of R.

(a) If R is commutative, show that R/I is commutative.

(b) If R has an identity, show that R/I also has an identity.

4. Determine whether each of the following polynomials is irreducible over the given field.

(a) $3x^4 + 5x^3 + 50x + 15$ over $\mathbb{Q}$.

(b) $x^2 + 7$ over $\mathbb{Q}$.

(c) $x^2 + 7$ over $\mathbb{C}$.